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SOME MODELS RELATING TO ENCOUNTERS BETWEEN SHIPS IN TRANSIT AND SUBMARINES ON PATROL

by

GIJSBERTUS E. VAN DOMSELAAR

15 DECEMBER 1980

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NORTH ATLANTIC TREATY ORGANIZATION

SACLANT ASW Research Centre
Viale San Bartolomeo 400, I-19026 San Bartolomeo (SP), Italy.

tel: national 0187 560940
international + 39 187 560940

telex: 271148 SACENT I

SOME MODELS RELATING TO ENCOUNTERS BETWEEN SHIPS IN TRANSIT AND SUBMARINES ON PATROL

by

Gijsbertus E. van Domselaar

15 December 1980

This memorandum has been prepared within the SACLANTCEN Systems Research Division as part of Project 06.

L.F. Whicker
Division Chief

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SOME MODELS RELATING TO ENCOUNTERS
BETWEEN SHIPS IN TRANSIT AND SUBMARINES ON PATROL

by

Gijsbertus E. van DOMSELAAR

ABSTRACT

Models are developed that describe encounters between convoys, or independent ships, and submarines. Two aspects of such encounters are considered in some detail:

- The rate at which encounters occur between submarines and their targets

- The time that a submarine remains within convoy defences when attempting to reach an attacking position.

INTRODUCTION

Recently in force employment studies several high level campaign models have been developed. These models mainly describe the interactions between Soviet submarines and NATO reinforcement and resupply shipping. Values of input parameters are required, which could either be determined by submodelling or simply be given. The choice highly depends on the sensitivity of the results to these input parameters and to the available information for developing submodels.

This memorandum describes submodels for two input parameters to higher models: the encounter rate between transitting NATO ships and a submarine on patrol, and the average time a submarine will spend inside the convoy defences when attempting to attack a convoy.

In section 1 a model for the encounter rate is developed. A well known encounter model appears as a special case. In section 2 a model is developed for the average time a submarine needs to pass a convoy screen.

1 ENCOUNTER RATE

1.1 Detection rate model

In this section, a model for the detection rate between shipping in transit (convoys or individual ships) and a submarine on patrol is derived. The final model is well known and similar results can be found in literature (for instance, see <1>). The derivation, however, is given as an introduction to the encounter model, for which model it will show its usefulness.

The configuration of the problem is illustrated in Fig. 1.

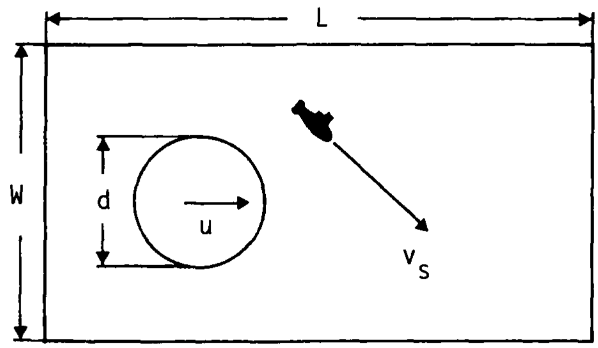

FIG. 1

The patrol area of the submarine has length L and width W. No boundary effects are considered, therefore it holds that d is much smaller than both W and L.
The targets (convoys or individual ships) are entering on a side W and the point of entrance is uniformly distributed over W. The targets are transitting through the area on a straight line with a speed u. The submarine is doing a random search in the area with a speed v_s.

The detection range of a submarine against a target is $\frac{d}{2}$.

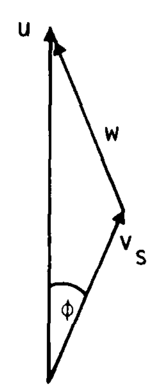

FIG. 2

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If ϕ is the angle between the target and submarine speed (see Fig. 2) then the speed of the target relative to the submarine becomes, using the cosine rule,

$$w = \sqrt{v_s^2 + u^2 - 2uv_s \cos\phi} \qquad \text{(Eq. 1)}$$

The submarine will detected the target when it is entering the detection circle (radius $\frac{d}{2}$) around the target.

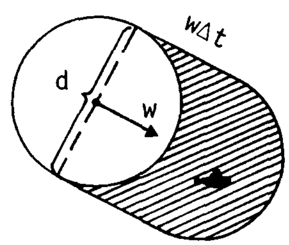

FIG. 3

During time interval Δt the submarine can only enter the detection circle when it is inside the shaded area of Fig. 3. It is assumed that the angle ϕ between u and v_s is uniformly distributed over the interval $[0, 2\pi]$. Therefore the probability that the submarine detects the target during time interval Δt , given the submarine is within the angle between ϕ and $\phi + d\phi$, becomes

$$\frac{dw}{LW} \frac{d\phi}{2\pi} \Delta t. \qquad \text{(Eq. 2)}$$

The rate per Δt that the submarine detects a target which is inside the patrol area then becomes

$$\frac{1}{2\pi} \frac{d}{LW} \int_0^{2\pi} w \, d\phi \, .$$

If the targets are entering the area with a rate of μ per Δt , then at any moment the number of targets inside the area is:

$$\mu \frac{L}{u} \, . \qquad \text{(Eq. 3)}$$

The rate, at which a submarine will detect the targets in the area per Δt, becomes

$$\varepsilon = \mu \; \frac{L}{u} \, \frac{1}{2\pi} \, \frac{d}{LW} \int_0^{2\pi} w \; d\phi \; .$$

Using the relations $\cos\phi = -\cos(\pi-\phi)$, $\cos\phi = 1 - 2\sin^2(\frac{\phi}{2})$ and Eq. 1 the detection rate becomes

$$\varepsilon = \mu \; \frac{d}{W} \; \frac{u+v_s}{u} \, \frac{2}{\pi} \int_0^{\pi/2} \sqrt{1 - \frac{4uv_s}{(u+v_s)^2} \sin^2 \phi} \; d\phi \quad ,$$

$$\varepsilon = \mu \; \frac{d}{W} \; \frac{u+v_s}{u} \, \frac{2}{\pi} \, E \left(\frac{2\sqrt{uv_s}}{u+v_s}, \; \frac{\pi}{2} \right), \qquad \text{(Eq. 4)}$$

where $E\left(\frac{2\sqrt{uv_s}}{u+v_s}, \; \frac{\pi}{2}\right)$ is the complete elliptic integral of the second kind.

1.1.1 Examples

Two special cases, in which the targets are convoys, are derived and one of them is compared with Eq. 4 in a numerical example.

1) The submarine's search speed equals the convoy speed, i.e. $u = v_s$. Since $E(1, \frac{\pi}{2}) = 1$, it follows that the detection rate becomes:

$$\varepsilon_1 = \frac{4}{\pi} \; \mu \; \frac{d}{W} \; .$$

2) Suppose that the submarine is stationary somewhere in its patrol area, i.e. $v_s = 0$. In that case $k = 0$ and knowing that $E(0, \frac{\pi}{2}) = \frac{\pi}{2}$, it yields the trivial detection rate

$$-\varepsilon_2 = \mu \; \frac{d}{W} \quad . \qquad \text{(Eq. 5)}$$

The expression for ε_2 has often been used as an approximation of the detection rate. For an example, the values of ε (Eq. 4) and ε_2 have been compared. The following input values have been taken:

$u = 15$ kn
$d = 200$ n.mi
$W = 300$ n.mi
$\mu = 0.33$ per day.

Figure 4 shows the graph of ε for different values of the submarine search speed v_s and the graph of ε_2. The figure demonstrates the increasing detection rate for increasing submarine speed.

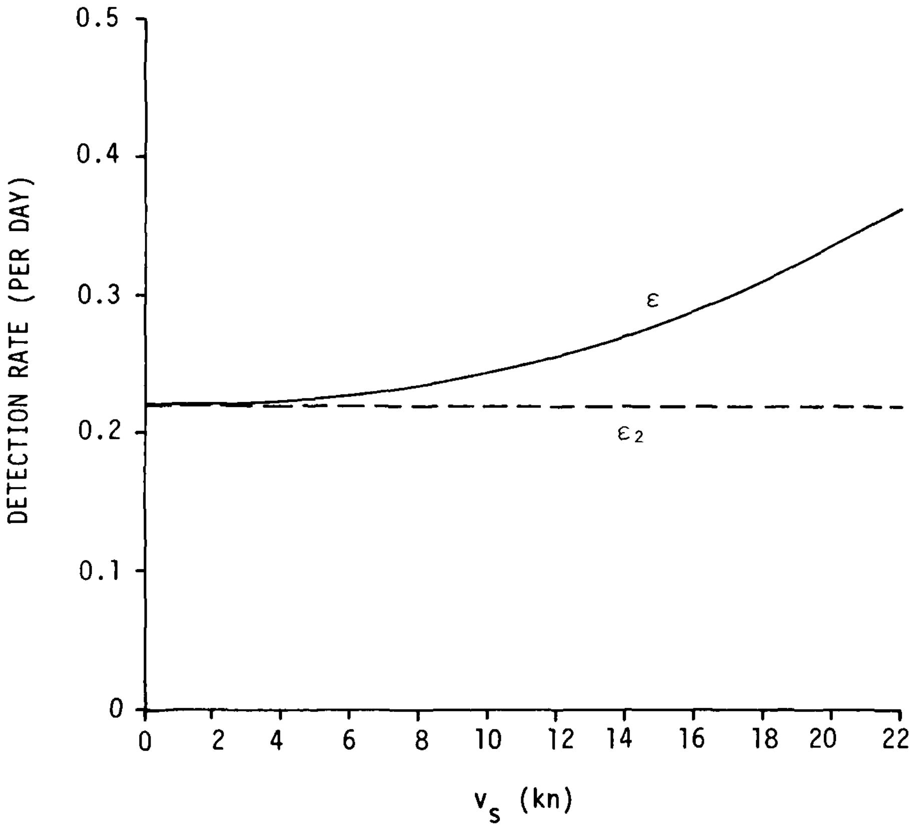

FIG. 4

1.2 Encounter rate model

After a detection a submarine will not always encounter the target. Because of speed limitations it is often not able to do so. But also in the case when the submarine is able to encounter a target, it will not necessarily try to intercept the target. When the submarine is confronted with plenty of targets it could select only the high value ones.

However, in this section it is assumed that the submarine will always aim for the target when it can be intercepted. The submarine will change its course such that the relative speed is towards the target and that it will change its search speed v_s to a closing speed v_c .

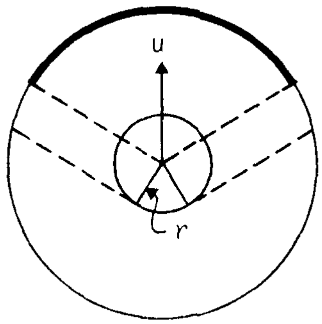

FIG. 5

When the closing speed of the submarine is superior to the speed of the target, the target will always be encountered after a detection and the encounter rate equals the detection rate of Eq. 4:

$$\lambda = \varepsilon \quad .$$

When the submarine's closing speed is lower than the target's speed, the submarine will only be able to encounter the target when it is entering the detection circle within the well known limited angle of approach:

$$- \arcsin(\frac{v_c}{u}) < \psi < \arcsin(\frac{v_c}{u}) \;, \qquad \text{(Eq. 6)}$$

where ψ is the angle between u and w .

This angle is illustrated in Fig. 5 with the thick line on the edge of the circle. Within this angle the submarine is able to reach the centre of the target. However in reality the submarine does not need to reach the centre of the target but only to come within a certain range of the centre of the target in order to fire its torpedoes or short range missiles. This range will be called the weapon firing range r . From Fig. 5 it can be seen that in this case the limited angle of approach becomes

$$- \psi^* < \psi < \psi^* \quad ,$$

where

$$\boxed{\psi^* = \arcsin\left(\frac{v_c}{u}\right) + \mathrm{arctg}\left(\frac{2r}{d}\right)} \qquad \text{(Eq. 7)}$$

Figure 6 shows that only when the submarine is within the shaded area during time interval Δt , will it be able to encounter the target.

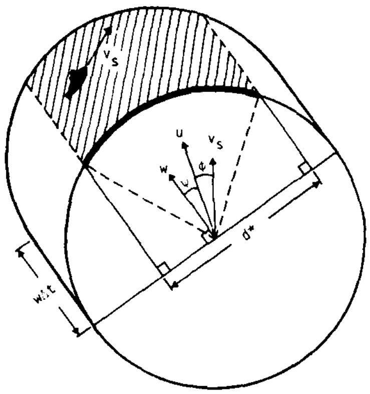

FIG. 6

Again assuming that the angle ϕ between u and v_s is uniformly distributed over $[0,2\pi]$, the encounter rate becomes

$$\lambda = \frac{\mu}{2\pi} \; \frac{1}{uW} \int_0^{2\pi} d^* \, w \, d\phi \; . \qquad \text{(Eq. 8)}$$

Obviously the value of d^* depends on the value of the angle ψ . In order to calculate d^* three cases have to be separated. In Appendix A the encounter rates for the three cases are derived. Here we will confine ourselves to the results.

1) If it holds that (see Fig. 6)

$$0 \leq \arcsin\left(\frac{v_s}{u}\right) < \frac{\pi}{2} - \psi^* \quad ,$$

then

$$\boxed{\lambda_1 = \mu \frac{d}{W} \sin \psi^*} \qquad \text{(Eq. 9)}$$

2)

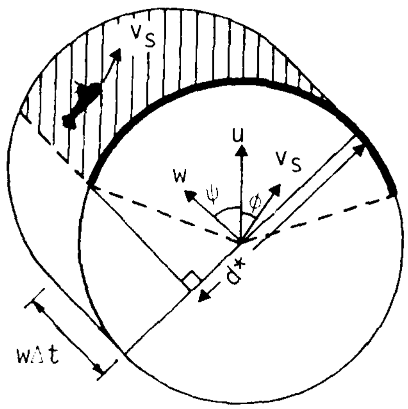

FIG. 7

If it holds that (see Fig. 7)

$$\arcsin\left(\frac{v_s}{u}\right) \geq \frac{\pi}{2} - \psi^* \quad \text{and} \quad \psi^* < \frac{\pi}{2} \quad ,$$

then

$$\boxed{\lambda_2 = \frac{\mu}{\pi}\frac{d}{W}\left[\left\{\pi - \frac{\phi_2 - \phi_1}{2}\sin\psi^* + \frac{u+v_s}{u}\left\{E\left(\frac{2\sqrt{uv_s}}{u+v_s}, \frac{\pi-\phi_1}{2}\right) - E\left(\frac{2\sqrt{uv_s}}{u+v_s}, \frac{\pi-\phi_2}{2}\right)\right\}\right\}\right]} \qquad \text{(Eq. 10)}$$

where E(.,.) is the elliptic integral of the second kind, and ϕ_1 and ϕ_2 are solutions of the quadratic equation:

$$\cos\left(\frac{\pi}{2} - \psi^*\right) = \frac{u - v_s \cos\phi}{w} \quad .$$

3)

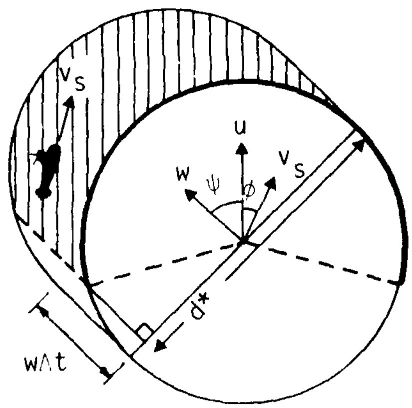

FIG. 8

If it holds that (see Fig. 8)

$$\psi^* \geqslant \frac{\pi}{2}$$

then

$$\boxed{\lambda_3 = \frac{\mu}{\pi}\frac{d}{W}\left[\frac{\phi_2-\phi_1}{2}\sin\psi^* + \frac{u+v_s}{u}\left\{2E\left(\frac{2\sqrt{uv_s}}{u+v_s},\frac{\pi}{2}\right) - E\left(\frac{2\sqrt{uv_s}}{u+v_s},\frac{\pi-\phi_1}{2}\right) + E\left(\frac{2\sqrt{uv_s}}{u+v_s},\frac{\pi-\phi_2}{2}\right)\right\}\right],} \qquad \text{(Eq. 11)}$$

where ϕ_1 and ϕ_2 are solutions of the quadratic equation

$$\cos(\psi^* - \frac{\pi}{2}) = \frac{u-v_s\cos\phi}{W} .$$

Note that at the boundaries

$$\arcsin(\frac{v_s}{u}) = \frac{\pi}{2} - \psi^*$$

and

$$\psi^* = \frac{\pi}{2}$$

it holds that $\lambda_1 = \lambda_2$ and $\lambda_2 = \lambda_3$, respectively.

It has been said that, when the submarine closing speed is superior to the target speed, the encounter rate equals the detection rate. But it can be verified that at the boundary

$$\arcsin(\frac{v_c}{u}) = \frac{\pi}{2}, \qquad \text{i.e. } v_c = u,$$

the encounter rate λ_3 does not equal the detection rate. The reason for this deviation is illustrated in Fig. 9, in which figure $u = v_c = v_s$ and the limited angle of approach is

$$\psi^* = \frac{\pi}{2} + \arctan(\frac{2r}{d}) .$$

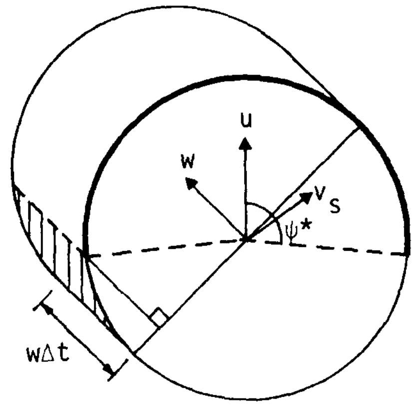

FIG. 9

If the submarine is in the shaded area of Fig. 9 during time interval Δt, it will detect the target. But it will not be able to encounter the target, because it is entering the detection circle outside the limited angle of approach.

The deviation at $u = v_c$ will also appear in a numerical example in Sect. 1.2.2.

1.2.1 Special case

A special case of the encounter model can be derived when the weapon firing range is taken $r = 0$.

Since $\mathrm{arctg}\,(\frac{2r}{d}) = 0$, the limiting angle of approach simply becomes

$$\psi^* = \arcsin(\frac{v_c}{u}) \quad . \qquad \text{(Eq. 12)}$$

Now, only two cases have to be separated:

1) If it holds that

$$0 \leqslant \arcsin(\frac{v_s}{u}) < \frac{\pi}{2} - \arcsin(\frac{v_c}{u})$$

then

$$\lambda_1 = \mu \; \frac{d}{W} \; \frac{v_c}{u} \quad . \qquad \text{(Eq. 13)}$$

This expression for the encounter rate is well known and often used to describe the encounter rate between a submarine on patrol and convoys or ships in transit. But it should be noted that the expression is only valid for the given restrictions.

2) If it holds that

$$\frac{\pi}{2} - \arcsin\left(\frac{v_c}{u}\right) \leqq \arcsin\left(\frac{v_s}{u}\right) \leqslant \frac{\pi}{2}$$

then

$$\lambda_2 = \frac{\mu}{\pi} \frac{d}{W} \left[\frac{v_c}{u} \left\{ \pi - \frac{\phi_2 - \phi_1}{2} \right\} + \right.$$

$$\left. \frac{u+v_s}{u} \left\{ E\left(\frac{2\sqrt{uv_s}}{u+v_s}, \frac{\pi-\phi_1}{2}\right) - E\left(\frac{2\sqrt{uv_s}}{u+v_s}, \frac{\pi-\phi_2}{2}\right) \right\} \right], \qquad \text{(Eq. 14)}$$

where

$$\phi_1 = \arccos\left(\frac{u}{v_s} - \frac{v_c^2}{uv_s} + \frac{v_c}{uv_s}\sqrt{v_s^2 + v_c^2 - u^2}\right),$$

$$\phi_2 = \arccos\left(\frac{u}{v_s} - \frac{v_c^2}{uv_s} - \frac{v_c}{uv_s}\sqrt{v_s^2 + v_c^2 - u^2}\right).$$

An example of this special case is given in the next section.

1.2.2 Example

In this section an application of the encounter model Eq. 9 to Eq. 11, where $r > 0$, and an application of the model Eqs. 13 and 14, where $r = 0$ are given. Both use the same inputs as to the numerical example in section 1.1.1.

In the first application it is assumed that the configuration of the convoy is such that the submarine needs to be 14 n.mi from the centre of the convoy in order to fire its torpedoes. In both examples it has been assumed that the search speed of the submarine equals the closing speed. The results are given in Fig. 10.

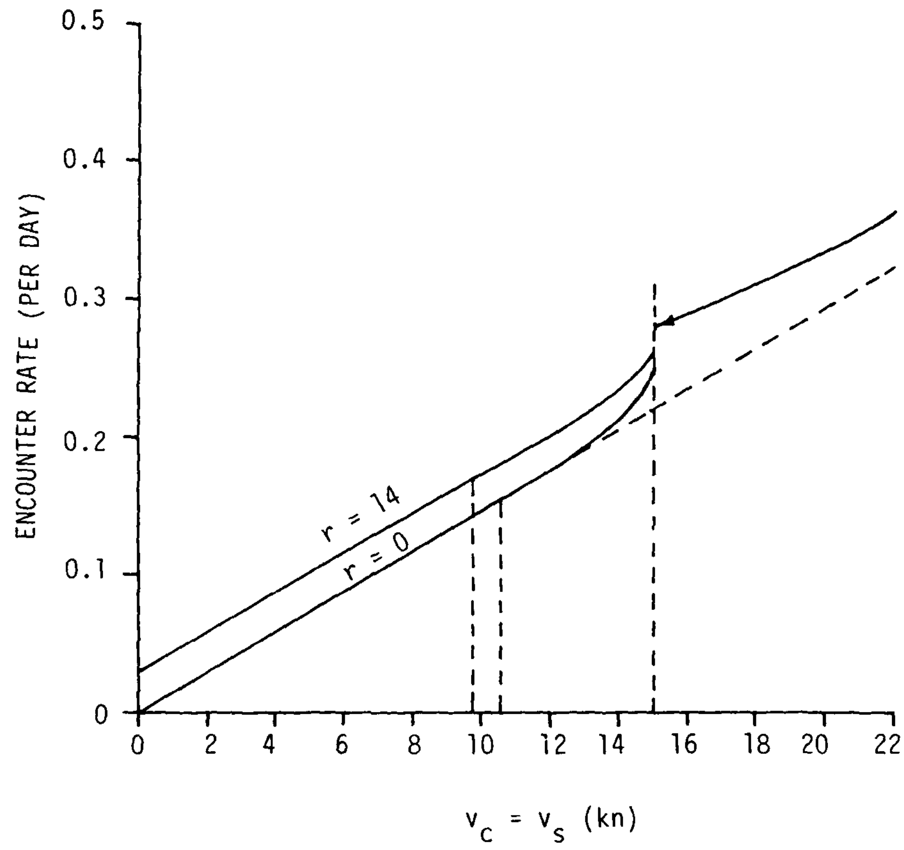

FIG. 10

Note the previously mentioned deviation at $u = v_s = v_c = 15$ kn. When the submarine speed is superior to the convoy speed, in both cases the encounter rate equals the same detection rate (see also Fig. 4).

2 TIME TO PASS A CONVOY SCREEN

2.1 Model

When a submarine has detected a defended convoy, it has to pass an anti-submarine screen before it is able to make an attack. Certain high level models describing the effectiveness of a convoy screen require the time a

submarine needs to pass a screen as an input parameter. In this section a model is presented for the average time a submarine will spend inside a convoy screen.

After a submarine has detected a convoy within the limited angle of approach, ψ^*, it will change its course and speed such that the relative speed, $\bar{w}$, is aimed at the centre of the convoy. Then, $\bar{\psi}$ is the angle between $\bar{w}$ and u. The configuration is given in Fig. 11.

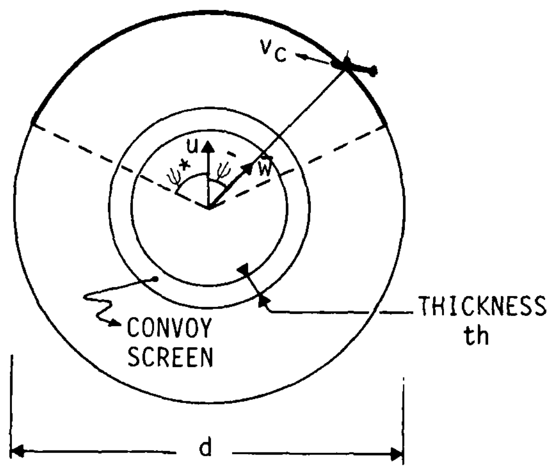

FIG. 11

It is assumed that the convoy screen is a circular band around the convoy, with a thickness th. Figure 11 shows that the time required to pass the screen simply becomes

$$\tau = \frac{th}{\bar{w}} \ . \qquad \text{(Eq. 15)}$$

However, to find the distribution of the angle $\bar{\psi}$ is less simple. In Appendix C the structure of a model for τ is given, which takes into account the relation between the angle $\bar{\psi}$ and the submarine search speed v_s.

Here, the point of entrance at the detection circle within the limited angle of approach ψ^* is assumed to be independent of the search speed. Suppose that the position of the submarine is uniformly distributed on a line ahead of the convoy, perpendicular to the transit direction of the convoy (see Fig. 12).

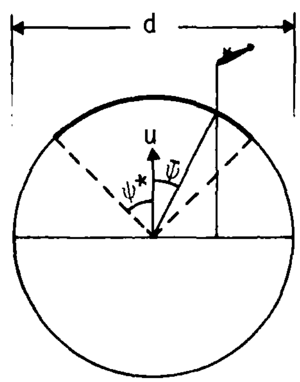

FIG. 12

This assumption is valid when the submarine search speed v_s is relatively small compared to the convoy speed u, but may also be valid when the submarine has some a priori information about the arrival of the convoy and stations itself in front of the convoy. In these circumstances, the probability density function (p.d.f.) of $\bar{\psi}$ can be derived as:

$$g(\bar{\psi}) = \frac{1}{2}\cos\bar{\psi} \ , \quad -\frac{\pi}{2} \leq \bar{\psi} \leq \frac{\pi}{2} \ . \qquad \text{(Eq. 16)}$$

Note that $\bar{\psi}$ is a different random variable from ψ in the previous sections.

Since the submarine will only aim for the convoy when it enters the limited angle of approach, the p.d.f. of $\bar{\psi}$ is required, given the submarine enters the detection circle within the limited angle of approach. This becomes:

$$\bar{g}(\bar{\psi}) = \frac{\cos\bar{\psi}}{2g^*} \ , \qquad -\psi^* \leq \bar{\psi} \leq \psi^* \ , \qquad \text{(Eq. 17)}$$

where

$$g^* = \int_{-\psi^*}^{\psi^*} \frac{1}{2}\cos\bar{\psi}\, d\bar{\psi} \ .$$

Since the submarine is aiming at the centre of the convoy it holds that

$$\psi^* = \arcsin\left(\frac{v_c}{u}\right) \ , \qquad v_c < u \ .$$

If the closing speed $v_c > 0$, then the p.d.f. becomes

$$\bar{g}(\bar{\psi}) = \frac{1}{2}\,\frac{u}{v_c}\cos\bar{\psi}, \ -\arcsin\left(\frac{v_c}{u}\right) \leq \bar{\psi} \leq \arcsin\left(\frac{v_c}{u}\right) \qquad \text{(Eq. 18)}$$

Applying the cosine rule to Fig. 11 gives

$$v_c^{\,2} = \bar{w}^2 + u^2 - 2u\bar{w}\cos\bar{\psi} \ .$$

From the two roots of this equation, the submarine will take the one which minimizes the time spent inside the screen. Therefore

$$\bar{w} = u\cos\bar{\psi} + \sqrt{v_c^{\,2} - u^2\sin^2\bar{\psi}} \ . \qquad \text{(Eq. 19)}$$

Then the average time a submarine will spend inside the screen, using the symmetry between $\bar{\psi}$ and $-\bar{\psi}$, becomes

$$\tau = 2 \int_{0}^{\psi^*} \frac{th}{\bar{w}} \, \bar{g}(\bar{\psi}) \, d\bar{\psi} \; .$$

Using the Eqs. 18 and 19 and the transformation $x = \sin(\bar{\psi})$ it can be found that

$$\tau = \frac{th}{u^2 - v_c^2} \frac{u}{v_c} \int_{0}^{\frac{v_c}{u}} \left\{ u \sqrt{1 - x^2} - v_c \sqrt{1 - \left(\frac{u}{v_c}\right)^2 x^2} \right\} dx \, ,$$

$$\boxed{\tau = \frac{th}{2} \frac{u}{u^2 - v_c^2} \left\{ \sqrt{1 - \left(\frac{v_c}{u}\right)^2} + \frac{u}{v_c} \arcsin\left(\frac{v_c}{u}\right) - \frac{v_c}{u} \frac{\pi}{2} \right\} ,} \qquad \text{(Eq. 20)}$$

where $v_c > 0$ and $v_c < u$.

In order to complete this section a general model should have been developed for the case in which the submarine's closing speed is superior to the convoy speed ($v_c > u$). However, such a model cannot be developed because the time the submarine spends inside the convoy screen is highly dependent on the tactics of the submarine. Because of its speed superiority the submarine is in principle able to approach from any direction. For instance after the detection it could move in front of the convoy and wait. Thus, each type of tactics requires a different speci.ic model to be developed.

2.2 Example

In Fig. 13, Eq. 20 has been applied to a convoy speed u of 15 kn and a screen thickness of 50 n.mi. The figure denotes the average time the submarine will spend inside the screen for different closing speeds v_c.

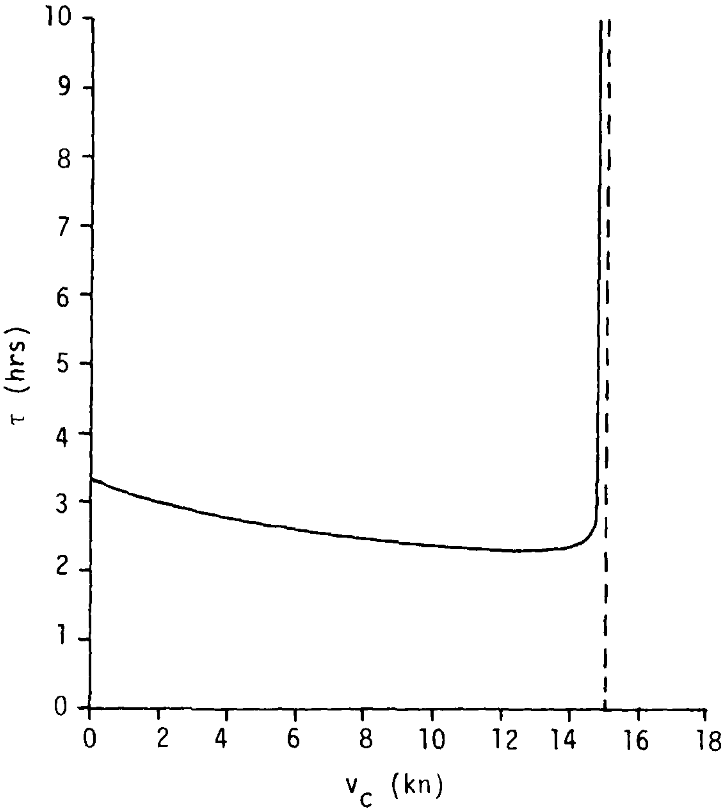

FIG. 13

The example shows that the parameter τ is rather insensitive to the closing speed of the submarine.

Note the asymptotic value of τ in the case when the submarines closing speed approaches the convoy speed. This theoretically means that, in the case $\bar{\psi} = \frac{\pi}{2}$, a submarine trying to encounter the convoy, will stay inside the screen for ever (see also Fig. 14).

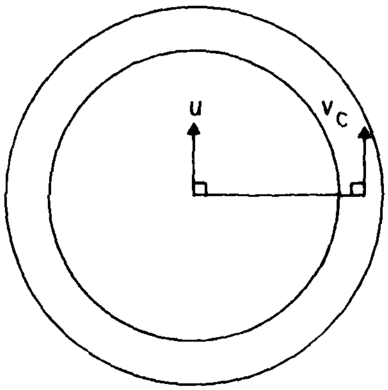

FIG. 14

CONCLUSIONS

In this report models have been developed relating to encounters between transitting convoys or independent ships and submarines on patrol. A general model for the encounter rate is developed. The model takes into account that a submarine does not necessarily move towards the centre of the target to make an encounter: to reach weapon firing range is a sufficient requirement. The effect of weapon firing range on the encounter rate is shown in an example.

A model is given for the average time a submarine will spend inside the convoy screen when it tries to make an encounter. An example indicates that this parameter is rather insensitive to the submarine's closing speed.

The models presented in this report can be regarded as sub-models of parameters of "high level" attrition models. They could either be used to calculate directly the values of these parameters or serve as a background for the understanding of them in the high level models.

REFERENCE

1. KOOPMAN, B.O. Search and Screening, Operations Evaluation Group, Office of the Chief of Naval Operations, Washington, D.C., OEG Report No. 56, ATI 64 627 (1946).

APPENDICES

APPENDIX A

DERIVATION OF THE ENCOUNTER MODEL

In Fig. A.1 it can be seen that the following relations between ϕ and ψ hold:

$$\cos\psi = \frac{u - v_s \cos\phi}{w} , \qquad \text{(Eq. A.1)}$$

$$\sin\psi = \frac{v_s \sin\phi}{w} . \qquad \text{(Eq. A.2)}$$

Moreover it holds that $\psi \leqslant \arcsin\left(\frac{v_s}{u}\right)$, all ϕ . (Eq. A.3)

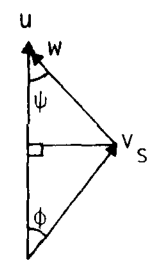

FIG. A.1

In order to calculate the encounter rate λ , the value of d^* has to be determined. For this purpose three cases have to be separated.

1) If $\arcsin\left(\frac{v_s}{u}\right) < \frac{\pi}{2} - \psi^*$, which means that the limited area of approach (ψ^*) lies inside the detection area, then

$$d^* = \frac{d}{2}\sin(\psi^* - \psi) + \frac{d}{2}\sin(\psi^* + \psi)$$

$$= d \sin\psi^* \cos\psi . \qquad \text{(Eq. A.4)}$$

(See also Fig. 6)

Then, using Eqs. 8 and A.1 and the symmetry of ϕ in the intervals $[0, \pi]$ and $[\pi, 2\pi]$, the encounter rate becomes:

$$\lambda_1 = \frac{\mu}{2\pi}\ \frac{1}{uW}\ 2d \sin\psi^* \int_0^{\pi} (u - v_s \cos\phi)\, d\phi \quad ,$$

$$\lambda_1 = \mu \frac{d}{W} \sin\psi^* \ . \qquad \text{(Eq. A.5)}$$

2) If $\arcsin\left(\frac{v_s}{u}\right) \geqslant \frac{\pi}{2} - \psi^*$ and $\psi^* < \frac{\pi}{2}$ (see Fig. 7) ,

then

$$d^* = \begin{cases} d \sin\psi^* \cos\psi \quad , & \text{if } 0 \leqslant \psi < \frac{\pi}{2} - \psi^* \\ \frac{d}{2}\sin(\psi^* - \psi) + \frac{d}{2} \quad , & \text{if } \frac{\pi}{2} - \psi^* \leqslant \psi \leqslant \arcsin\left(\frac{v_s}{u}\right) \ . \end{cases}$$

Using the relations (A.1) and (A.2), it can be found that

$$d^* = \begin{cases} d \sin\psi^* \dfrac{u - v_s \cos\phi}{w}, & \text{if } 0 \leqslant \phi < \phi_1 \text{ and } \phi_2 < \phi \leqslant \pi \\ \dfrac{d}{2}\left[1 + \sin\psi^* \dfrac{u - v_s \cos\phi}{w} - \dfrac{v_s}{w} \cos\psi^* \sin\phi\right], & \\ \text{if } \phi_1 \leqslant \phi \leqslant \phi_2 \ , & \end{cases} \qquad \text{(Eq. A.6)}$$

where ϕ_1 and ϕ_2 are solutions of the quadratic equation

$$\cos\left(\frac{\pi}{2} - \psi^*\right) = \frac{u - v_s \cos\phi}{w} \quad .$$

The solutions ϕ_1 and ϕ_2 are given in Appendix B.1.
The following relations hold:

$$\cos\phi_2 - \cos\phi_1 = -\operatorname{ctg}\left(\frac{\pi}{2} - \psi^*\right) \left\{\sin\phi_2 - \sin\phi_1\right\} , \qquad \text{(Eq. A.7)}$$

$$\int_{\phi_1}^{\phi_2} w d\phi = 2(u+v_s)\left\{E\left(\frac{2\sqrt{uv_s}}{u+v_s}, \frac{\pi-\phi_1}{2}\right)-E\left(\frac{2\sqrt{uv_s}}{u+v_s}, \frac{\pi-\phi_2}{2}\right)\right\}.$$

(Eq. A.8)

Relation (A.7) is proved in Appendix B.2.

With the aid of Eqs. A.6 to A.8 the encounter rate of Eq. 8 can be written as λ_2 , presented in Eq. 10.

3) If $\psi^* \geqslant \frac{\pi}{2}$

then

$$d^* = \begin{cases} d & \text{, if } 0 \leqslant \psi < \psi^* - \frac{\pi}{2} \\ \frac{d}{2}\sin(\psi^*-\psi)+\frac{d}{2} & \text{, if } \psi^*-\frac{\pi}{2} \leqslant \psi \leqslant \arcsin\left(\frac{v_s}{u}\right) \end{cases} .$$

Using the relations (A.1) and (A.2), it can be found that

$$d^* = \begin{cases} d & \text{, if } 0\leqslant\phi<\phi_1 \text{ and } \phi_2<\phi\leqslant\pi \\ \frac{d}{2}[1+\sin\psi^* \frac{u-v_s \cos\phi}{w} - \frac{v_s}{w}\cos\psi^* \sin\phi] & \text{, if } \phi_1\leqslant\phi\leqslant\phi_2 \end{cases} ,$$

(Eq. A.9)

where ϕ_1 and ϕ_2 are solutions of the quadratic equation

$$\cos(\psi^*-\frac{\pi}{2}) = \frac{u-v_s \cos\phi}{w} .$$

See Appendix B.1 for the solutions ϕ_1 and ϕ_2.

Applying Eqs. A.7 to A.9 to Eq. 8, the encounter rate λ_3 can be written as presented in Eq. 11.

APPENDIX B.1

THE SOLUTIONS ϕ_1 AND ϕ_2

The equation

$$\cos\psi = \frac{u - v_s \cos\phi}{w},$$

where $w = \sqrt{u^2 + v_s^2 - 2uv_s \cos\phi}$ and ψ has a given value, can be written as

$$v_s^2 \cos^2\phi + 2\,uv_s(\cos^2\psi - 1)\cos\phi + u^2(1 - \cos^2\psi) - v_s^2 \cos^2\psi = 0 .$$

This quadratic equation has the roots

$$\phi_1 = \arccos\left(\frac{u}{v_s}\sin^2\psi + \frac{\cos\psi}{v_s}\sqrt{v_s^2 - u^2\sin^2\psi}\right),$$

$$\phi_2 = \arccos\left(\frac{u}{v_s}\sin^2\psi - \frac{\cos\psi}{v_s}\sqrt{v_s^2 - u^2\sin^2\psi}\right).$$

Note that the condition $v_s^2 - u^2 \sin^2\psi \geqslant 0$ is always valid (see Eq. A.3).

APPENDIX B.2

THE PROOF OF RELATION (Eq. A.7)

Applying the sine rule to Fig. B.2.1 gives the relation

$$\frac{u}{\sin(\pi-\psi-\phi)} = \frac{v_s}{\sin\psi}, \quad \text{where } \phi = \phi_1, \phi_2 .$$

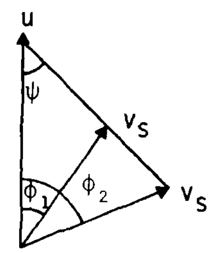

FIG. B.2.1

This relation can be written as

$$\frac{u}{v_s}\sin\psi = \sin\psi\cos\phi + \sin\phi\cos\psi .$$

Substituting the solutions ϕ_1 and ϕ_2 of Appendix B.1 into this equation gives

$$\sin\phi_1 = \sin\psi\cos\psi - \frac{u}{v_s}\sin\psi\sqrt{\frac{v_s^2}{u^2} - \sin^2\psi},$$

$$\sin\phi_2 = \sin\psi\cos\psi + \frac{u}{v_s}\sin\psi\sqrt{\frac{v_s^2}{u^2} - \sin^2\psi}.$$

Using the relation

$$\cos\phi_2 - \cos\phi_1 = -2\frac{u}{v_s}\cos\psi\sqrt{\frac{v_s^2}{u^2} - \sin^2\psi},$$

it follows that

$$\cos\phi_2 - \cos\phi_1 = -\operatorname{ctg}\psi\,(\sin\phi_2 - \sin\phi_1).$$

Equation A.7 is found by taking $\psi = \frac{\pi}{2} - \psi^*$.

APPENDIX C

THE STRUCTURE OF A MODEL FOR τ

In this Appendix the structure of a model for parameter τ is derived which is more general than the model in Sect. 2. At the moment a submarine enters the detection circle, the convoy is sailing with a relative speed w towards the stationary submarine. This relative speed is determined by the speeds of the submarine (v_s) and convoy (u).

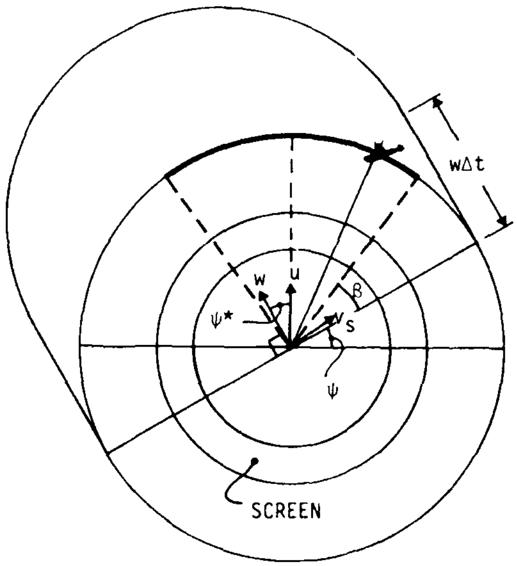

FIG. C.1

In Fig. C.1 it can be seen that the point of entrance at the detection circle is determined by the angle β. Similar to the derivation of the p.d.f. in Eq. 17 it can be found that the p.d.f. of the random variable β becomes

$$g(\beta) = \frac{\sin\beta}{2}, \qquad 0 \le \beta \le \pi . \qquad \text{(Eq. C.1)}$$

When the submarine enters the detection circle within the limited angle of approach, ψ^*, it changes its course and speed (from v_s to v_c) such that, similar to Eq. 19, the relative speed w becomes:

$$\bar{w} = u \sin\beta + \sqrt{v_c^2 - u^2\cos^2\beta} . \qquad \text{(Eq. C.2)}$$

In order to calculate the average time a submarine spends inside the screen, two cases have to be separated. The separation is similar to the separation made for the special case in Sect. 1.2.1.

1) $\arcsin(\frac{v_s}{u}) < \frac{\pi}{2} = \psi^*$, where $\psi^* = \arcsin(\frac{v_c}{u})$.

This means that the maximum amplitude of ψ less than $\frac{\pi}{2} - \psi^*$ (see also Fig. C.1).

In this case the p.d.f. of the random variable β, given that the submarine enters the detection circle within the limited angle of approach, becomes

$$g_1(\beta) = \frac{\sin\beta}{2g_1^*}, \qquad \frac{\pi}{2} - \psi - \psi^* \leq \beta \leq \frac{\pi}{2} - \psi + \psi^*, \qquad \text{(Eq. C.3)}$$

where

$$g_1^* = \int_{\pi/2-\psi-\psi^*}^{\pi/2-\psi+\psi^*} \frac{\sin\beta}{2}\, d\beta = \frac{\sin(\psi+\psi^*) - \sin(\psi-\psi^*)}{2}. \qquad \text{(Eq. C.4)}$$

Then

$$\tau = 2\int_0^{\pi}\int_{\pi/2-\psi-\psi^*}^{\pi/2-\psi+\psi^*} \frac{th}{w}\, g_1(\beta)\, f(\phi)\, d\beta\, d\phi, \qquad \text{(Eq. C.5)}$$

where $f(\phi)$ is the p.d.f. of ϕ, which is the angle between u and v_s. As before it could be assumed that ϕ is uniformly distributed, i.e.

$$f(\phi) = \frac{1}{2\pi}, \qquad 0 \leq \phi \leq 2\pi.$$

The relations between ϕ and ψ are given by

$$\cos\psi = \frac{u - v_s\cos\phi}{w} \quad \text{and} \quad \sin\psi = \frac{v_s\sin\phi}{w},$$

where w is as in Eq. 1.

$$\text{2)} \quad \arcsin\left(\frac{v_s}{u}\right) \geqq \frac{\pi}{2} - \psi^*.$$

Similarly as in Sect. 1.2 the integrals are divided by the cases

$$\psi \leqq \frac{\pi}{2} - \psi^* \quad \text{and} \quad \psi \geqq \frac{\pi}{2} - \psi^*.$$

In case $\psi \leq \frac{\pi}{2} - \psi^*$ Eq. C.4 still holds.

In case $\psi \geq \frac{\pi}{2} - \psi^*$, the p.d.f. of β, given that the submarine will encounter the convoy, becomes

$$g_2(\beta) = \frac{\sin\beta}{2g_2^*} \qquad 0 \leq \beta \leq \frac{\pi}{2} - \psi + \psi^*$$

where

$$g_2^* = \int_0^{\pi/2-\psi+\psi^*} \frac{\sin\beta}{2}\, dB = \frac{1 - \sin(\psi-\psi^*)}{2}$$

The average time a submarine will spend inside the screen, given the submarine will encounter the convoy, becomes

$$\tau = 2\int_0^{\phi_1}\int_{\pi/2-\psi-\psi^*}^{\pi/2-\psi+\psi^*} \frac{th}{\bar{w}}\, g_1(\beta)\, f(\phi)\, d\beta\, d\phi \;+$$

$$2\int_{\phi_2}^{\pi}\int_{\pi/2-\psi-\psi^*}^{\pi/2-\psi+\psi^*} \frac{th}{\bar{w}}\, g_1(\beta)\, f(\phi)\, d\beta\, d\phi \;+ \qquad \text{(Eq. C.6)}$$

$$2\int_{\phi_1}^{\phi_2}\int_{0}^{\pi/2-\psi+\psi^*} \frac{th}{\bar{w}}\, g_2(\beta)\, f(\phi)\, d\beta\, d\phi\;,$$

where $f(\phi)$ is as defined after Eq. C.5, and ϕ_1 and ϕ_2 are the solutions of the equation

$$\cos\left(\frac{\pi}{2} - \psi^*\right) = \frac{u - v_s\cos\phi}{w}\;, \quad \text{(see Appendix B.1).}$$

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